BOUNDED SETS OF NODES IN TRANSFINITE TREES

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ABSTRACT

In this note we give a recursive characterization of bounded sets of nodes in transfinite trees.

The purpose of this note is to give a recursive characterization—suggested by Rabin's theory of automata on infinite trees [1]—of bounded sets of nodes in transfinite trees. A careful study of some of the proofs in this theory shows that a recurrent use is made of some "principle" concerning subsets of a tree which have a finite intersection with every path in the tree. This principle is made explicit, generalized, and independently proved here as Theorem 2, since it is believed that it is of some importance and may have other applications too.

We begin by some notation and definitions. A *tree* is a partially ordered non-empty set T such that the following holds:

1) For every $x \in T$, the set $P(x) = \{y : y < x\}$ is a well-ordered set. Its order-type α will be called the *rank* of x. The set of all x with rank α will be denoted by T_α ($T = \cup T_\alpha$).

2) For every limit $\alpha \geqq 0$, if $x \in T_\alpha$, and $y \in T_\alpha$ and $P(x) = P(y)$ then $x = y$.

Taking $\alpha = 0$ in condition 2, we get that there is a *unique* $x_0 \in T$ such that $P(x_0) = \emptyset$; x_0 will be called *the root* of T. The other members of T are *nodes* of T.

For x a node in T we denote by T_x the subtree of T with root x; that is: $T_x = \{y : y \in T, y \geqq x\}$. T_x^- is $T_x - \{x\}$. A tree is *locally finite* if T_α is finite for every α.

A *path* in T is a totally ordered (and hence well-ordered) subset of T, which contains the root of T. An *α-path* is a path of length α. T is an *α-tree* if it has no β-path for $\beta > \alpha$, but for every $\gamma < \alpha$ has a path of length greater than γ.

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Let T be an α-tree, $T_x \subseteq T$, and L a path in T_x; L will be called *principal* if it is cofinal with α; a subset of L is *bounded* if it is not cofinal with α; finally L^- will denote $L - \{x\}$.

We now state the following:

THEOREM 1. *Every locally finite α-tree has an α-path.*

For non-limit α the claim is obvious. For limit α this is a standard generalization of König's lemma, which may be proved by using any "maximality" principle such as Zorn's lemma or Robinson's valuation lemma ([2], p. 13).*

DEFINITION. Let T be any α-tree, α a limit ordinal, and $T' \subseteq T$ a set of nodes of T. We say that T' has the *bounded-intersection* property (relative to T) if the intersection of T' with every *α-path* of T is bounded.

For $\alpha = \omega$ the condition states that the intersection of T' with every path of T is finite. This obviously does not entail that T' itself is finite, as may be seen by taking for T the "full binary tree" (i.e., the set of all finite words on $\{0,1\}$, ordered by the "intial" relation: $x \leqq y$ iff there is some z such that $xz = y$), and for T' the set $\{0^k1;\ k \geqq 1\}$. We want to characterize this property by some more "closed" conditions.

For $T' \subseteq T$, we define recusively a set of nodes of T, the *associated-set* of T', $H(T')$, as follows: $H(T')$ is the least set that contains all nodes x which satisfy the following condition:

(C) For every principal path $L \subseteq T_x$, $L^- \cap T' = \emptyset$ or there is node $y \in L^-$ such that $y \in H(T')$.

In particular $H(T')$ contains all nodes x for which $T_x^- \cap T' = \emptyset$.

THEOREM 2. *Let T be a locally finite tree, x_0 its root, and $T' \subseteq T$. Then T' has the bounded-intersection property if and only if $x_0 \in H(T')$.*

PROOF. Let T be a locally-finite α-tree, and T' a non-empty subset of T.

1) Suppose first that $x_0 \notin H(T')$ (in short, H); we show that T' does not have the bounded-intersection property. For $x \in T$, a path $L \subseteq T_x$ will be called *acceptable* if L is a principal path, $L^- \cap T' \neq \emptyset$, and $L^- \cap H = \emptyset$. For every $\rho < \alpha$ we define, by induction, a node $y_\rho \in T$ and an acceptable path $L_\rho \subseteq T_{y_\rho}$ such that

(1) (y_ρ) is an increasing sequence of nodes;

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* Or, more simply, the possibility of embedding every proper filter in an ultrafilter. I am indebted to the referee for this remark.

(2) For non-limit ρ, $y_\rho \in T'$.

Thus, the intersection of T' with the unique principal path containing all the y_ρ will not be bounded, and the claim will be proved. Since $x_0 \notin H$ there is an acceptable path $L \subseteq T$; we put $y_0 = x_0$, $L_0 = L$. Suppose y_ρ and L_ρ have been defined and satisfy conditions (1) and (2) for every $\rho < \beta$.

Assume first that β is a non-limit ordinal; then $L_{\beta-1}^- \cap T' \neq \emptyset$, $L_{\beta-1}^- \cap H = \emptyset$. Let y_β be the first node in $L_{\beta-1}^- \cap T'$; $y_\beta \notin H$ so there is an acceptable path $L \subseteq Ty_\rho$, take $L_\beta = L$.

For β a limit ordinal, let $T_\beta^* = \{z_1, \cdots, z_n\} \subseteq T_\beta$ be the set of all nodes in T_β which are contained in principal paths of T. ($T_\beta^* \neq \emptyset$ as assured in Theorem 1). For each $\rho < \beta$, L_ρ is a principal path so there is a unique $i(\rho)$, $1 \leqq i(\rho) \leqq n$ such that $z_{i(\rho)} \in L_\rho$ and $L_\rho \mid \beta = P(z_{i(\rho)})$ ($L_\rho \mid \beta$ is the initial subpath of L_ρ of length β). There is thus some $i_0, 1 \leqq i_0 \leqq n$, and a sequence of ordinals (ρ_δ) cofinal with β such that $L_{\rho_\delta} \mid \beta = P(z_{i_0})$. Since $y_{\rho_\delta} \in L_{\rho_\delta}$ and the sequence of nodes (y_ρ) is linearly ordered, we conclude that $y_\rho \in P(z_{i_0})$ for every $\rho < \beta$; also (since $z_{i_0} \in L_{\rho_\delta}$) $z_{i_0} \notin H$, so there is an acceptable path $L \subseteq T_{z i_0}$. If we put $y_\beta = z_{i_0}$, $L_\beta = L$, conditions (1) and (2) are clearly satisfied and the induction step is complete.

2) For every $\rho \geqq 0$ we define inductively the following sets of nodes (L is a variable ranging over principal paths):

$$H_\rho = \{x : (L)\ (L \subseteq T_x \Rightarrow L^- \cap T' = \emptyset \vee (\exists \beta)\ (\exists y)\ (\beta < \rho \wedge y \in L^- \cap H_\beta)\}$$

Clearly $H(T') = \cup H_\rho$. Suppose now $x_0 \in H(\equiv H(T'))$; then $x_0 \in H_{\alpha_0}$, for some α_0. We show that for every path $L \subseteq T$, $L \cap T'$ is bounded. Take any α-path $L_0 \subseteq T$. If $L_0^- \cap T' = \emptyset$, we are finished. Otherwise, there is some $x_1 \in L_0^-$ and $\alpha_1 < \alpha_0$, such that $x_1 \in H_{\alpha_1}$. Let $L_1 = L_0 \mid T_{x_1}$ (the restriction of L_0 on T_{x_1}). If $L_1^- \cap T' = \emptyset$, we are again finished; in the other case, there is some $x_2 \in L_1^-$ and $\alpha_2 < \alpha_1$ with $x_2 \in H_{\alpha_2}$. Continuing this process we get an increasing sequence of nodes $x_0 < x_1 < x_2 < \cdots <$, and a *decreasing* sequence of ordinals $\alpha_0 > \alpha_1 > \alpha_2 > \cdots$ such that $x_i \in H_{\alpha_i}$, so that the process must stop after a *finite* number of steps, say at α_n. This may happen either because $L_n^- \cap T' = \emptyset$, or because $\alpha_n = 0$ so that for every principal $L \subseteq T_{x_n}$, $L^- \cap T' = \emptyset$, and again $L_n^- \cap T' = \emptyset$; in either case $L_0 \cap T'$ is bounded.

Q.E.D.

REMARKS

1) If we replace condition (C) above by the following one:

(C′) $T_x^- \cap T' = \emptyset$ or there is a node $y \in T_x^-$ such that $y \in H(T')$—then one side of the equivalence in Theorem 2 remains true; namely: if T' has the bounded-intersection property then $x_0 \in H(T')$.

The proof goes as follows: Suppose $x_0 \notin H$; then $T_{x_0} \cap T' \neq \emptyset$ and $T^-_{x_0} \cap H = \emptyset$. Define inductively an increasing sequence of nodes $(y_\rho)_{\rho<\alpha}$, in which $y_{\rho+1} \in T'$ (and by assumption, $y_\rho \notin H$), as follows: $y_0 = x_0$; suppose y_ρ has been defined; $y_\rho \notin H$ so $T^-_{y_\rho} \cap T' \neq \emptyset$; take as $y_{\rho+1}$ any element in this set; if y_ρ has been defined for every $\rho < \beta$, β a limit ordinal, then take as y_β the least upper bound of (y_ρ). The sequence (y_ρ) defines thus a unique L for which $L \cap T'$ is unbounded.

The converse however, is obviously no more true as may be seen by taking for T any locally finite ω-tree (which is not linearly ordered), and for T' any ω-path in T.

2) On the other hand, replacing (C) by

(C″) $T_x^- \cap T = \emptyset$ or for every principal $L \subseteq T_x$ there is some $y \in L^-$ such that $T_y^- \cap T' = \emptyset$

reverses the situation since, clearly, if $x_0 \in H$ then T' has the bounded-intersection property, while the converse is again false, as may be seen by the example given after Theorem 1, in which T' has the bounded-intersection property, but $x_0 \notin H$.

REFRENCES

1. M. O. Rabin, *Decidability of second-order theories and automata on infinite trees*, Trans. Amer. Math. Soc. **141** (1969), 1–35.

2. A. Robinson. *Introduction to Model Theory and to the Metamathematics of Algebra*, North-Holland, Amsterdam, 1965.

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